On the semisimplicity of braid group representations in $\mathbb{C}$-linear, semisimple braided tensor categories.

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Let B_k denote the k-strand braid group. Given a braided $\mathbb{C}$-linear tensor category $\mathcal{C}$, any object X gives algebra homomorphisms

$$\Phi_k : \mathbb{C}B_k \to \mathrm{End}_{\mathcal{C}}(X^{\otimes k}).$$

Question.

If $\mathcal{C}$ is a semisimple category then are the braid representations semisimple?

Notation: X has "Property S" if the answer is yes (for all k).

Motivation

Knowing the braid reps are *a priori* semisimple helps in the classification of ribbon categories with fusion rules of Lie type.

Classification of tensor categories with fusion rules of Lie type:

No braiding

- $SL(2)$: Kerler 1992
- $SL(N)$: Kazhdan and Wenzl 1993
- $SO(3)$: Etingof and Ostrik 2018

With braiding

- $O(N)$ and $Sp(N)$: Tuba and Wenzl 2003
- $SO(N), N \neq 4$: C. 2020

The results state that any semisimple (ribbon) category with the fusion rules of G (or a related truncated fusion ring) is equivalent to a quantum group category (possibly up to a twist of the associator and/or modifying the braiding.)

Unitary braided tensor categories have semisimple braid representations.

Unitarizability of quantum group categories.

$\mathcal{C}(\mathfrak{g}, q, l)$: the semisimple ribbon category constructed from the $\text{Rep}(U_q\mathfrak{g})$ with q^2 a primitive l-th root of 1.

m: ratio of long roots to short roots in $\mathfrak{g}$, squared

- Wenzl 1998: When $m|l$ and $q = e^{\pi i/l}$, $\mathcal{C}(\mathfrak{g}, q, l)$ is unitarizable.

Corollary

These categories (and their Galois conjugates) have Property S.

Categories not included: Type BC with l odd (Rowell 2005), Type D with l odd, Type E_6 with $3 \nmid l$, Type E_7 with $2 \nmid l$.

More examples:

- Categories with Property F (image of braid group is finite)
  - Includes the Drinfel'd centers of pointed categories (Etingof-Rowell-Witherspoon 2008)
  - More generally, all known weakly integral categories.
- Symmetric tensor categories
  - More interesting: if X is a self dual object, are the representations of the Brauer algebra $\mathrm{Br}_k(\dim X)$ in $\mathrm{End}_{\mathcal{C}}(X^{\otimes k})$ semisimple?

Categories generated by braid morphisms.

If the maps $\Phi_k : \mathbb{C}B_k \to \mathrm{End}_{\mathcal{C}}(X^{\otimes k})$ are onto, then X has Property S.

Quantum group examples: variants of "Quantum Schur duality"

- For $SL(N)$, X the vector rep: Jimbo 1986, Kazhdan-Wenzl
- For $\mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}, O(N)$, X the vector rep: Wenzl 1990, Tuba-Wenzl
- Many cases when q not a root of 1, including G_2, $X = 7$ dim'l rep: Lehrer-Zhang 2006

Non-example: $SO(2n)$-type quantum group categories, X the vector rep'n. In addition to the braids, an additional generator in $\mathrm{End}_{\mathcal{C}}(X^{\otimes n})$ is required (corresponding to the Pfaffian.)

$SO(2n)$-type categories.

The $SO(2n)$-type fusion rings are the subrings of $\mathfrak{so}(2n)$-type fusion rings spanned by simples with integer highest weights (no spin rep'ns).

- $\mathcal{C}$: semisimple ribbon category with $SO(2n)$-type fusion rules.
- X: object corresponding to the vector rep'n.

Thm (C. 2020): Classification of $SO(2n)$-type categories, $n > 2$.

Up to a $\mathbb{Z}_2$-cocycle twist of the associator and modification of the braiding, $\mathcal{C}$ is ribbon equivalent to a quantum group category.

Corollary of proof: The braid representations in $\mathrm{End}_{\mathcal{C}}(X^{\otimes k})$ are semisimple.

Ideas in the proof: reduction to the study of braid representations.

$\mathcal{C}$: arbitrary semisimple ribbon category with fusion rules of type $SO(2n)$.

- Kazhdan-Wenzl reconstruction: since $\mathcal{C}$ is $\mathbb{Z}_2$-graded the category is determined (up to a $\mathbb{Z}_2$-cocycle twist) by the tower of algebras

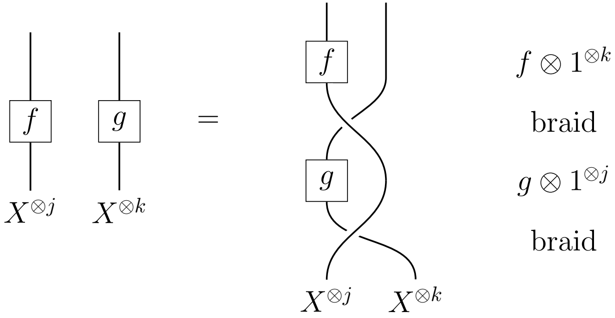
$$\cdots \to \mathrm{End}_{\mathcal{C}}(X^{\otimes k}) \xrightarrow{-\otimes 1} \mathrm{End}_{\mathcal{C}}(X^{\otimes k+1}) \to \dots$$

  and the tensor product maps

$$\mathrm{End}_{\mathcal{C}}(X^{\otimes k}) \times \mathrm{End}_{\mathcal{C}}(X^{\otimes j}) \to \mathrm{End}_{\mathcal{C}}(X^{\otimes k+j}).$$

- The algebras and inclusions are determined by the fusion rules.

- In the braided setting, the braid morphsims determine the tensor product of morphisms:



- Hence the category is determined by the braid representations together with the fusion rules.

Computing braid representations from $SO(2n)$-type fusion rules + the fusion rules.

Step 1: The braid element $c_{X,X} =$  always acts semisimply and since

$$X^{\otimes 2} = \mathbf{1} \oplus Y \oplus Z$$

there exist $q, r \in \mathbb{C}^{\times}$ so $c_{X,X}$ has eigenvalues $(r^{-1}, q, -q^{-1})$.

Step 2: Prove that the braid representations depend only on q (this is "braid rigidity" in the language of Martirosyan and Wenzl (2020)).

- This step is made complicated by having to rule out non-semisimple braid representations and relies heavily on the fusion rules.
- We use the q-Jucys Murphy approach to compute the twist values on simple objects while simultaneously obtaining restrictions on r and q.
- Finally one proves the braid representations are uniquely determined by q. The resulting representations are semisimple.

Final remarks on the Property S question.

A "no" answer means Property S is a nontrivial obstruction to the properties examined earlier (unitarizability, property F, generation by braids).

A "yes" answer to the semisimplicity question will be useful for classification techniques based on the computation of braid representations.

Thanks for listening!