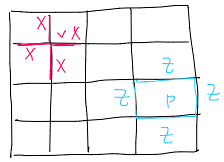
Interpreting toric code as an instance of TV TQFT

Thursday, June 3, 2021 11:56 PM

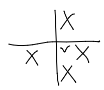
Toric code:



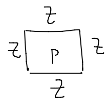
$\mathcal{H}$ = Hilbert space = $\bigotimes_{\text{edges}} \mathbb{C}^2$ $\left(= \bigotimes_{\text{edges}} \mathbb{C}G \right.$ for $\left. G = \mathbb{Z}_2 \right)$

$\rightsquigarrow$ $\underset{\text{surface}}{(\Sigma}, \underset{\text{cell decomp}}{\Delta)} = (T^2, \ N\times N \text{ lattice})$

vertex operators: A_v



plaquette operators: B_p



(Note: more convenient to use $\tilde{B}_p := \frac{1}{2}(I + B_p)$ which is a projection, and $\tilde{B}_p|\psi\rangle = |\psi\rangle \Updownarrow B_p|\psi\rangle = |\psi\rangle$)

Hamiltonian: $\sum_v I - A_v + \sum_p I - B_p = H$

ground states: correspond to $|\psi\rangle \in \mathcal{H}$ s.t. $A_v|\psi\rangle = B_p|\psi\rangle = |\psi\rangle$

- dimension $= 4 = \# H^1(T^2, \mathbb{Z}_2)$

(in general $= \#\left(\{\text{homs } \pi_1(\Sigma) \to G\}/_{\text{conjugation by } G}\right)$)

(in particular, the dim. of ground states is independent of N, actually a topological invariant of Σ).

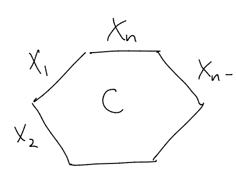
Turaev-Viro theory for the lattice on a torus: (assigns a vector space $Z_{TV}(\Sigma)$ to the torus)

Given (Σ, Δ) surface w/cell-decomp. and $\mathcal{C} = \mathrm{Rep}\, \mathbb{Z}_2$

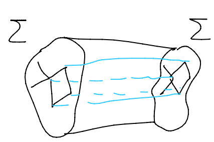
Let $\mathcal{H}^{TV}(\Sigma, \Delta) := \bigoplus_{\substack{\ell \text{ simple labelings of edges} \\ \text{by objects in Rep } \mathbb{Z}_2}} \bigotimes_{C \text{ face}} H(C, \ell)$

where $H(C, \ell) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X_1 \otimes \cdots \otimes X_n)$
$= \langle X_1, \dots X_n \rangle$

when



Consider $\Sigma \times [0,1]$ (cobordism $\Sigma \Rightarrow \Sigma$)



w/ cell decomp $\Delta \times I$

Then the TV theory assigns to this a linear map (using the def'n given by Itai)

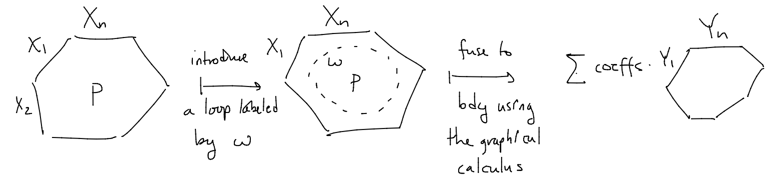
$$Z^{TV}(\Sigma \times I, \Delta \times I) : H^{TV}(\Sigma \times I) \longrightarrow H^{TV}(\Sigma \times I)$$ (in fact a projection)

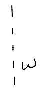
and then $Z^{TV}(\Sigma) :=$ image of this projection.

Thm: this image is independent of the original cell decomp Δ, ie only depends on the topology of Σ.

Important technical result: This projection corresponds w/ the operator

$\prod_p B_p^{TV}$ where B_p^{TV} acts locally at a plaquette by



where  denotes an edge labeled by the object $\omega = \frac{1}{\mathcal{D}^2} \sum_{X \in \mathrm{Irr}(\mathcal{C})} \dim_q(X)\, X$ $\left(= \frac{1}{2}(\mathbb{1} \oplus \text{sign}) \text{ when } \mathcal{C} = \mathrm{Rep}\, \mathbb{Z}_2 \right)$

think $\mathbb{1} = |+\rangle\langle +|$
$\text{sign} = |-\rangle\langle -|$

Thm (Balsam-Kirillov, Thm 4.1)

$Z^{TV}(\Sigma) \cong$ ground state space of Kitaev lattice model

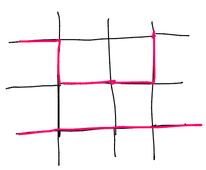
based on $\mathcal{C} = \mathrm{Rep}\, \mathbb{Z}_2$ — based on $G = \mathbb{Z}_2$.

Idea: The ground state of just the A_v's reproduces $H^{TV}(\Sigma, \Delta^*)$

Then the B_p condition $B_p|\psi\rangle = |\psi\rangle$ $\forall p$ translates to $\left(\prod_p B_p^{TV}\right)|\psi\rangle = |\psi\rangle$.

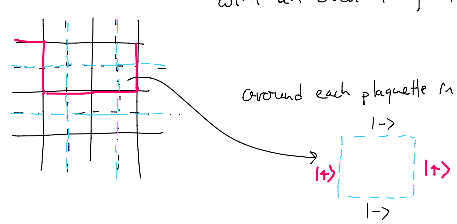
So the ground state for toric code (A_v's & B_p's) is the image of $\left(\prod_p B_p^{TV}\right) \subseteq H^{TV}(\Sigma, \Delta^*)$, which is $Z^{TV}(\Sigma)$.

Toric code example: • The conditions $A_v|\psi\rangle = |\psi\rangle$ mean $|\psi\rangle$ is a lin. combination of states w/ $|+\rangle$'s and $|-\rangle$'s on every edge with an even # of $|-\rangle$'s around each vertex.

$|\psi\rangle = \sum$ 

red = $|-\rangle$
black = $|+\rangle$

the dual graph



around each plaquette in the dual graph we have simple labeling of edges s.t.

$H^{TV}(C, \ell) = \langle X_1, X_2, X_3, X_4 \rangle \neq 0$

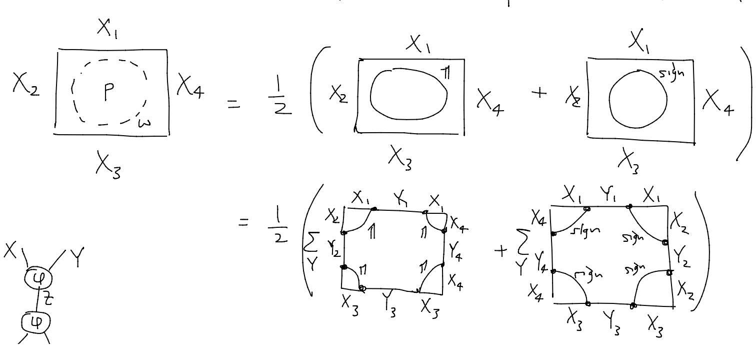
$(\Leftrightarrow X_1 \otimes X_2 \otimes X_3 \otimes X_4 = \mathbb{1}$

$\Leftrightarrow$ even # of $|-\rangle$'s).

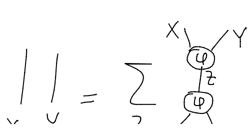
So $|\psi\rangle \in \bigoplus_{\ell \text{ simple label}} \bigotimes_{C \text{ in dual graph}} H^{TV}(C, \ell)$

• Now the condition $B_p|\psi\rangle = |\psi\rangle$ forall $|\psi\rangle$ corresponds to $B_p^{TV}|\psi\rangle = |\psi\rangle$ $\forall p$.

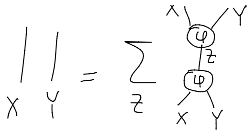
Examine B_p^{TV}:

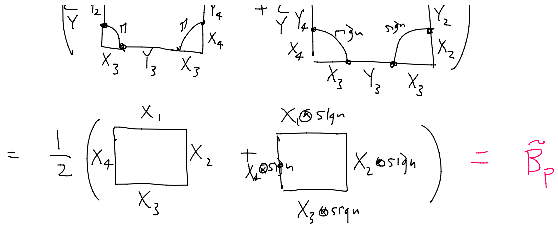


local relations from the:



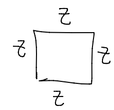
local relations from the graphical calculus for $\mathcal{C}$:





$= \tilde{B}_p$

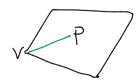
corresponds to



usual toric code plaquette operator

Excited states :

In Kitaev code, excitations are localized at sites $s = (v, p)$



$|\psi\rangle$ is an excitation localized at $s = (v, p)$ when

$$A_w |\psi\rangle = |\psi\rangle \qquad \forall w \neq v$$

$$B_q |\psi\rangle = |\psi\rangle \qquad \forall q \neq p$$

At every site there is an algebra of local operators

$$\{ I, A_v, B_p, A_v B_p \}$$

which generate an algebra which is a quotient of

$$D(\mathbb{C}\mathbb{Z}_2) = \mathbb{C}\mathbb{Z}_2 \otimes \mathrm{Fun}(\mathbb{Z}_2, \mathbb{C})$$

Drinfel'd double

$\mathrm{Rep}(D(\mathbb{C}\mathbb{Z}_2))$

The possible excitations at p are classified by irreps of $D(\mathbb{C}\mathbb{Z}_2)$ = objects of $\mathcal{Z}(\mathrm{Rep}\,\mathbb{Z}_2)$

(In the case of $D(\mathbb{C}\mathbb{Z}_2)$ there are 4 irreps

$\mathbb{1}$ (trivial) $\quad e \quad m \quad \psi$, tensor product rules : $e \otimes e = \mathbb{1}$, $m \otimes m = \mathbb{1}$, $\psi \otimes \psi = \mathbb{1}$, $e \otimes m = \psi$, ⋮)

Turaev-Viro : possible excitations are described by the category

$$Z^{TV}(S^1) = \text{Drinfel'd center of } \mathcal{C}.$$