3-2-1 TQFTs (once extended 2+1 TQFT)

Friday, June 4, 2021 10:28 AM

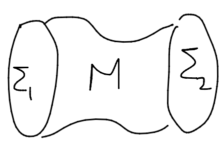
Recall: we've seen the Turaev-Viro TQFT as a 2+1 TQFT

It is a (symmetric monoidal) functor between the (symmetric monoidal) categories

$$Z^{TV}: \mathrm{Cob}_{2,3} \longrightarrow \mathrm{Hilb}$$

objects: closed oriented surface Σ $\longmapsto Z^{TV}(\Sigma)$ Hilbert space

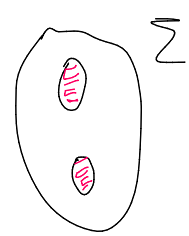
morphisms: 3d cobordism $M: \Sigma_1 \Rightarrow \Sigma_2$



$\longmapsto Z^{TV}(M): Z^{TV}(\Sigma_1) \to Z^{TV}(\Sigma_2)$ linear map.

Informally: $Z^{TV}(\Sigma_1)$ gives the info of the "ground states" for a local Hamiltonian on the surface Σ.

In order to get excited states, we need to consider $Z(\Sigma)$ where Σ is a surface w/boundary components (circles)



3-2-1 TQFTs (once-extended 2+1 dim'l TQFT)

A 3-2-1 TQFT is described by a (symmetric monoidal) 2-functor between 2-categories:

$$Z: \mathrm{Cob}_{1,3} \longrightarrow 2\mathrm{Hilb}$$

objects: $\otimes$-categories over $\mathbb{C}$

1-morphisms: (monoidal) functors

$Z : \mathrm{Cob}_{1,3} \longrightarrow 2\mathrm{Hilb}$

1-morphisms: (monoidal) functors

2-morphisms: (monoidal) natural transformations.

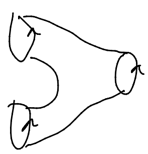
objects: closed oriented 1-mfds $B \longmapsto Z(B)$ a $\otimes$-category

$\emptyset \longmapsto \mathrm{Hilb}$ always

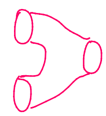
$S^1 \longmapsto \mathcal{D}$ some $\otimes$-category (must be braided)

$\mathcal{D} = Z(\mathcal{C})$ Drinfel'd center

1-morphisms: 2 dim'l cobordisms between 1-mfds



$\Sigma \longmapsto Z(\Sigma)$ functor

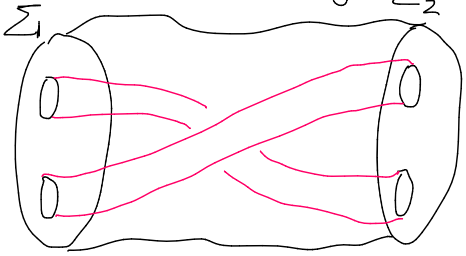
e.g. $Z($  $)$ = functor: $\mathcal{D} \boxtimes \mathcal{D} \to \mathcal{D}$

tensor product of categories

$\mathcal{D} \boxtimes \mathcal{D} = Z\left(\begin{smallmatrix} \circ \\ \circ \end{smallmatrix}\right)$, $\mathcal{D} = Z(\circ)$

2-morphisms: 3 dim'l cobordisms between surfaces w/boundary

Σ_1 Σ_2

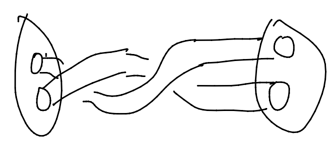


M

$M \longmapsto Z(M) : Z(\Sigma_1) \Rightarrow Z(\Sigma_2)$

natural transformation

Σ_1 Σ_1



$\partial M = \Sigma_1 \sqcup \Sigma_2$, which themselves have boundary.

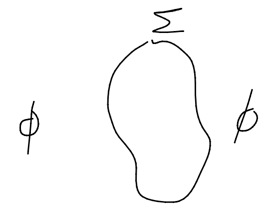
The "paths" of these boundary components can be viewed as embedded tubes in M, or framed ribbons.

Remarks: • A 3-2-1 TQFT "contains" a 2+1 TQFT.

A surface w/o boundary is a cobordism $\emptyset \Longrightarrow \emptyset$

Σ

A surface w/o boundary is a cobordism ...



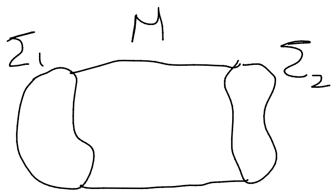
$\overset{Z}{\longmapsto}$ $Z(\Sigma): Z(\phi) \to Z(\phi)$

functor : $\mathrm{Hilb} \xrightarrow{F} \mathrm{Hilb}$

such a functor is completely determined by

$F(\mathbb{C})$ (ie choice of Hilbert space).

A cobordism between surfaces w/o boundary:



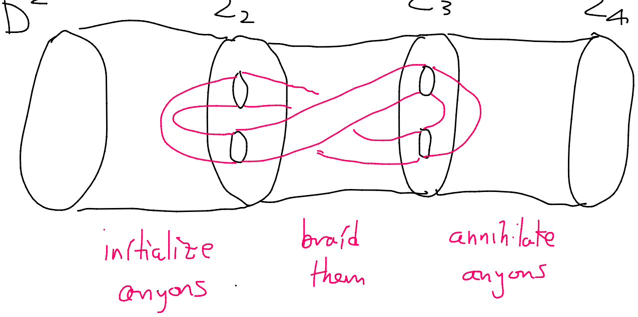
$\overset{Z}{\longmapsto}$ $Z(M)$: nat. transformation $Z(\Sigma_1) \Rightarrow Z(\Sigma_2)$

corresponds to a linear map

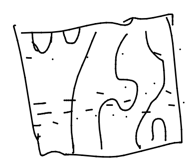
$F_1(\mathbb{C}) \to F_2(\mathbb{C})$.

- Example of computation w/ 3-2-1 TQFT:

$\Sigma_1 = D^2$ $\qquad \Sigma_2 \qquad \Sigma_3 \qquad \Sigma_4 = D_2$



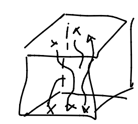
$\times = A \,|| - A^{-1} \,{}^{\cup}_{\cap}$

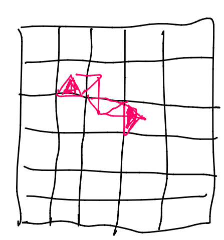


$\mathcal{C}$

$Z(\mathcal{C})$

Applying Z: get a linear map $Z(\Sigma_1) \to Z(\Sigma_4)$





- My intuition (no reference found): the embedded tubes correspond to the ribbon operators in the toric code

Fact: The Turaev-Viro TQFT can be extended to a 3-2-1 TQFT

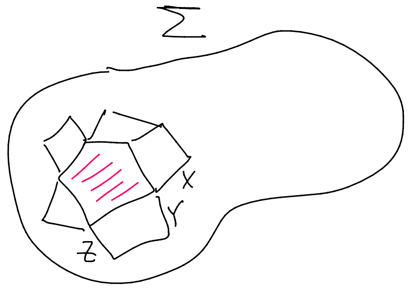
If the original 2+1 TV used the category $\mathcal{C}$, then the 3-2-1 theory assigns the category $Z(\mathcal{C})$ to the circle (Drinfel'd center)

$\cup$

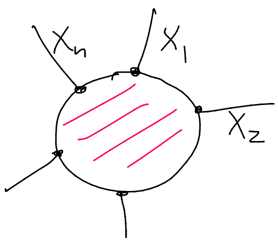
$Z(\mathcal{C})$ to the circle (Drinfel'd center)

Very rough idea:

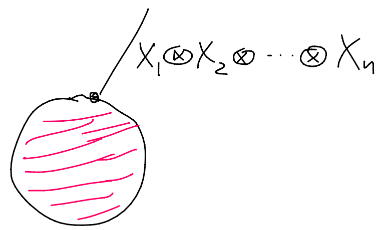
In 2+1 TV:



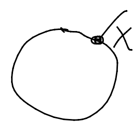
- edges are labeled by objects of $\mathcal{C}$
- If the surface has a boundary circle, then it looks like



using local relations in $\mathcal{C}$ (fusion of strings)

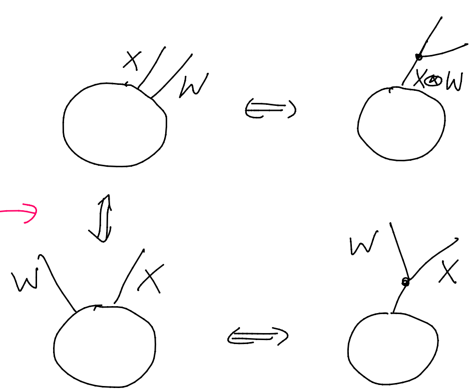


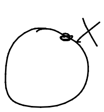
So the "category of boundary values" (what the 3-2-1 assigns to S^1)

is a circle w/ a labeled point  X object in $\mathcal{C}$

However, the local relations should allow

pull W around the circle



So if  is a valid boundary value, then X should come equipped w/

isomorphisms $X \otimes W \longrightarrow W \otimes X$ for any other object W.

The category whose objects are (object X of $\mathcal{C}$, isos $\gamma_{X,W}: X\otimes W \to W\otimes X$)
"half-braiding"

is exactly the Drinfel'd center of $\mathcal{C}$.

- To add "domain walls" and "boundary excitations" we should upgrade to a 3-2-1-0 TQFT (fully extended a la Lurie)

$$Z: \mathrm{Cob}_{0,3} \longrightarrow 3\,\mathrm{Hilb}$$

3-functor between 3-cats

objects: $\otimes$-cats $\mathcal{C}$

1-morphisms: bimodule categories ${}_{\mathcal{C}}M_{\mathcal{D}}$

2-morphisms: bimodule functors ${}_{\mathcal{C}}M_{\mathcal{D}} \to {}_{\mathcal{C}}N_{\mathcal{D}}$

3-morphisms: bimodule natural transformations

The additional "down to points" structure yields more operations for the topological quantum computer.

Domain walls & boundary excitations can be studied naively in the toric code

(perspective found in C. Delaney's thesis, still needs the language of bimodule categories).

Models for gapped boundaries and domain walls

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