The Kitaev lattice model & Turaev-Viro TQFT

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Reference:

KITAEV'S LATTICE MODEL AND TURAEV-VIRO TQFTS

BENJAMIN BALSAM AND ALEXANDER KIRILLOV, JR.

arXiv: 1206.3908

General idea of TQFTs / topological quantum computing:

Use algebraic (categorical data) to produce invariants of manifolds / a model of computation.

From a TQC perspective, the goal is to understand the algebraic data which results in a universal model of quantum computation.

- Examples: - toric code does not give universality. (Kitaev)
  - toric code w/ S_3 + "magic states" gives universality (Mochon)
  - toric code w/ "domain walls", "gapped boundaries", ... may give universality (?)
  - Property F conjecture (certain algebraic property of initial data ⟹ non-universality)

Another question: do these models support "natural" algorithms to solve nontrivial problems

- Example: approximating the Jones poly. of a link is BQP-complete, inspired by TQC

Roadmap of mathematical constructions (models) related to TQC

	toric code	$\subseteq$	Kitaev lattice model	$\subseteq$	Levin-Wen model string-net	$\subseteq$	Turaev-Viro TQFT	$\subseteq$	Reshetikhin Turaev TQFT (out of Witten's)
algebraic data:	$\mathbb{Z}_2$ or G finite group		H f.d. semisimple Hopf algebra		spherical $\otimes$-cat		spherical $\otimes$-cat		modular $\otimes$-cat (has braiding)

$$Z_{TV}(\mathcal{C}) = Z_{RT}(Z(\mathcal{C}))$$

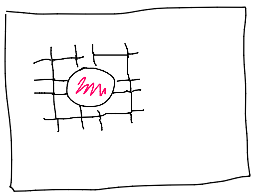
$Z(\mathcal{C})$ = Drinfel'd centers

$A \subseteq B$ means "A can be interpreted in terms of B" ie B generalizes A (often nontrivially)

Today: discuss connection of toric code w/ Turaev-Viro.

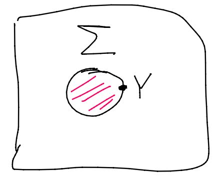
- in particular,

  excitations in the toric code $\longleftrightarrow$ "boundary values" in TV TQFT as a 3-2-1 TQFT.



$\{A_v^-, B_p^-, A_v^- B_p^-\}$

 local operators



Σ surface w/ bdy components

$Y \in Z(\mathcal{C})$

excitation = region of surface where the local Hamiltonian is not in a ground state

TV assigns a vector space

$Z_{TV}(\Sigma, Y)$

given a choice of "boundary value" on each boundary component.

$\mathcal{C} = \mathbb{Z}_2$

$Z(\mathcal{C}) = \mathrm{Rep}(D(G))$

Thm If the lattice model is defined using the group G, then the possible excitations are objects of $\mathrm{Rep}(D(G))$

$D(G) = \mathbb{C}G \otimes \mathrm{Fun}(G, \mathbb{C})$

quantum double, a Hopf algebra w/ R-matrix

$\{1, e, m, \psi\}$

$\mathrm{Rep}\, \mathbb{Z}_2 = \{1, \mathrm{sign}\}$

Thm If Z_{TV} is defined using the category $\mathcal{C}$, then the possible boundary values are objects of $Z(\mathcal{C})$

$Z(\mathcal{C})$ = Drinfeld double of $\mathcal{C}$, a modular $\otimes$-cat (braided in particular)

purely categorical construction

Note: $\mathrm{Rep}(D(G)) \simeq Z(\mathrm{Rep}(G))$ as categories.

(so these Thms are expressing the same thing, the TV version works for ANY spherical fusion category, not just Rep G).

Upshot: In TQC, anyons $\longleftrightarrow$ excitations $\longleftrightarrow$ bdy values in TV TQFT
(Kitaev-Levin-Wen)

$\longleftrightarrow$ objects of $Z(\mathcal{C})$ Drinfel'd center (braided $\otimes$-cat)

toric code: $\mathcal{C} = \mathrm{Rep}\, \mathbb{Z}_2$