Classification of ribbon categories with the fusion rules of $SO(N)$

Final defense

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Why ribbon categories?

Ribbon categories are highly structured algebraic objects, strongly motivated by low-dimensional topology, which permit topological reasoning in a graphical calculus.

"*In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.*"

–Hermann Weyl, *Invariants*, Duke Jour. Math, 1939.

We study ribbon categories algebraically but they arise in topological, analytic, geometric and physical contexts.

Quantum topology: algebra to topology

Quantum algebraic data provides local data for the construction of TQFTs, and in turn invariants of manifolds.

TQFT	input	dimension of invariant
Reshetikhin-Turaev (1991)	modular tensor cat	3
Turaev-Viro (1992)	spherical fusion cat	3
Crane-Yetter (1993)	ribbon cat	4
⋮		
Douglas-Reutter (2018)	spherical fusion 2-cat	4
Chaidez-Cotler-Cui (2020)	Hopf algebra	4

Theorem (Cobordism Hypothesis. Baez-Dolan, Lurie)

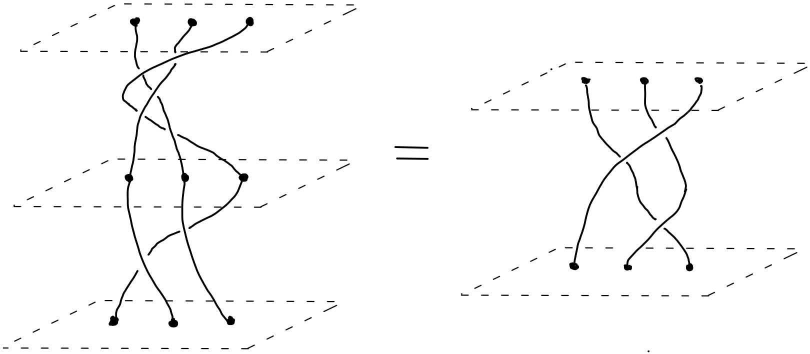
$$\left\{ \begin{array}{c} \textit{fully extended} \\ (n+1) - \textit{TQFTs} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{fully dualizable objects} \\ \textit{in a symmetric } (\infty, n)\textit{-category} \end{array} \right\}$$

Using topology for algebra: the graphical calculus

In order to analyse ribbon categories we go the other way, using topology to study algebraic objects and their representations.

Example

The n-strand braid group B_n, defined topologically.



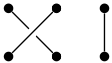
Generators and relations

Theorem (Artin, 1926)

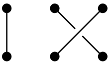
B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \textit{for } |i-j| \geq 2$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \textit{for } i = 1, 2, \ldots, n-1$$

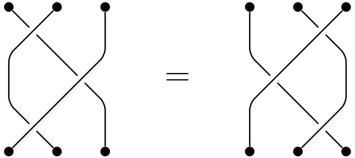
These are the *braid relations.* Having a short list of generators and relations helps to construct *representations* of B_n, as well as identify known objects as quotients of B_n (or its group algebra).



σ_1



σ_2

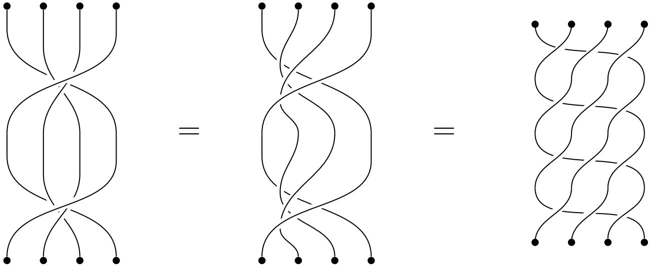


$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$

The full twist

On an elementary level our topological intuition can be used to study the algebraic structure of the braid group.

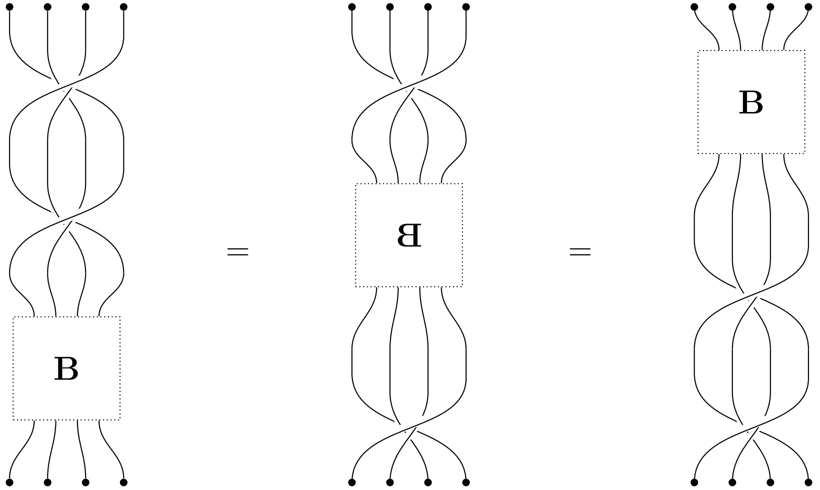
$\Delta_4^2 :=$



$$(\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2\sigma_3)^4$$

The full (counter clockwsise) twist on 4 strands

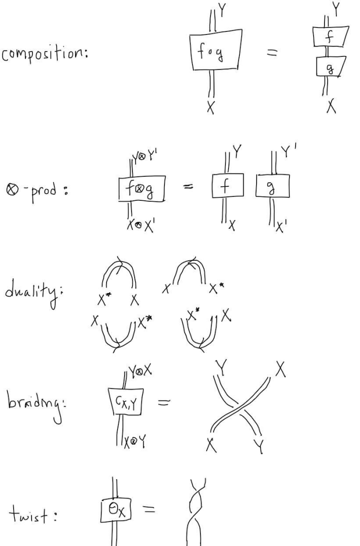
Full twist is central

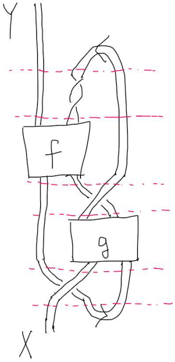


Δ_4^2 commutes with every braid on 4 strands.

What are the axioms of a ribbon category?

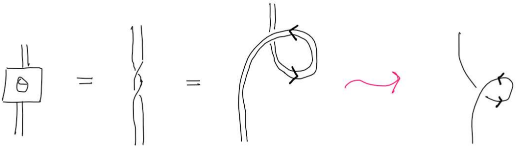
A *ribbon category* is a $\mathbb{C}$-linear semisimple monoidal category with compatible braiding, duality and twist structures.

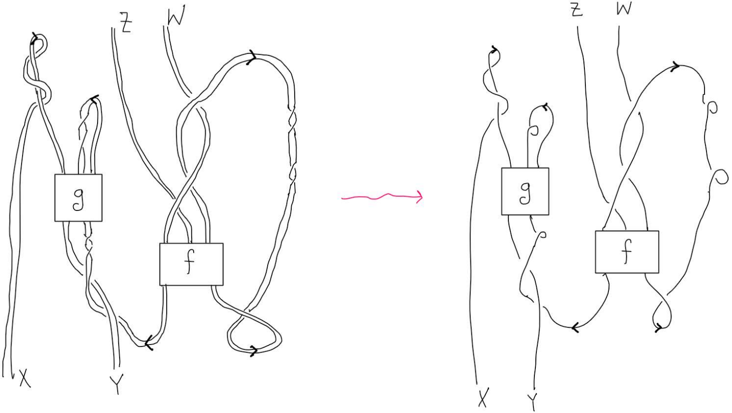




A morphism $X \to Y$.

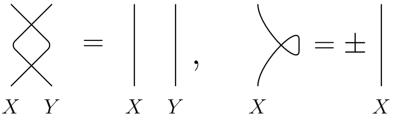
Blackboard framing



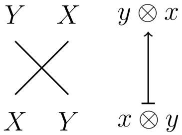


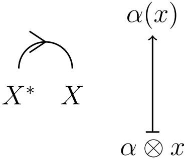
Where are ribbon categories?

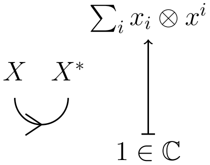
Symmetric tensor categories are everywhere, e.g. **Vec**, **Rep** G, combinatorial categories.



In **Vec**, X and Y are finite dimensional $\mathbb{C}$-vector spaces.





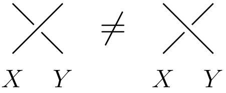


Theorem (Deligne, '02)

Any symmetric tensor category (subject to certain finiteness conditions) is equivalent to **Rep** G *or* **Rep** (G, ϵ).

Non-symmetric ribbon categories

$$c_{X,Y} \neq c_{Y,X}^{-1}$$



Drinfel'd-Jimbo quantum groups: $U_q\mathfrak{g}$ Hopf algebra

- For q not a root of 1: **Rep** $U_q\mathfrak{g}$ is semisimple with fusion rules of $\mathfrak{g}$
- For q a root of 1: **Rep** $U_q\mathfrak{g}$ is not semisimple, but we can extract a semisimple category $(\textbf{Rep } U_q\mathfrak{g})^{ss}$ using *tilting modules* (Andersen, '92).

Define **Rep** $SO(N)_q$ as the tensor subcategory of **Rep** $U_q\mathfrak{so}(N)$ or **Rep** $U_q\mathfrak{so}(N)^{ss}$ spanned by simples with integer highest weights (no spin reps).

The Grothendieck ring and fusion rules

The *Grothendieck ring* of a (semisimple) ribbon category $\mathcal{C}$ is generated by simple isotypes $\lambda \in \Gamma$, with relations

$$\lambda \otimes \mu = \sum_{\nu \in \Gamma} N_{\lambda,\mu}^{\nu} \nu$$

where $N_{\lambda,\mu}^{\nu}$ is the multiplicity of ν in $\lambda \otimes \mu$. $\mathrm{Gr}(\mathcal{C})$ is a $\mathbb{Z}$-*based ring*, equipped with *simple elements* as a $\mathbb{Z}$-basis.

$$\Gamma(\mathbf{Rep}\ \mathbb{Z}_2) = \{1, -1\}$$
$$\mathrm{Gr}(\mathbf{Rep}\ \mathbb{Z}_2) \cong \mathbb{Z}[\mathbb{Z}_2] \cong \mathbb{Z}[x]/(x^2 - 1).$$

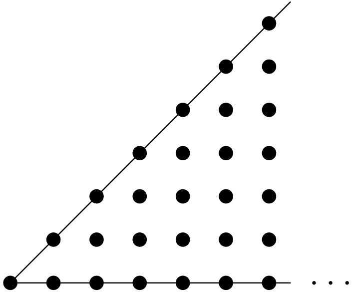
Up to monoidal equivalence, there are 2 ribbon categories with $\mathrm{Gr}(\mathcal{C}) \cong \mathrm{Gr}(\mathbf{Rep}\ \mathbb{Z}_2)$. Up to ribbon equivalence, there are 8.

$SO(N)$ fusion rules via highest weight

Finite dim irreps of $SO(N)$ are parametrized by their *highest weight* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, which must belong to the *dominant Weyl chamber*:

$$\Gamma(SO(2n+1)) = \{\lambda \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$$
$$\Gamma(SO(2n)) = \{\lambda \mid \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \geq 0\}.$$

The fusion rules are "generalized LR coefficients" and are given by classical formulas (e.g. Steinberg's rule). They define $\mathrm{Gr}(SO(N))$.

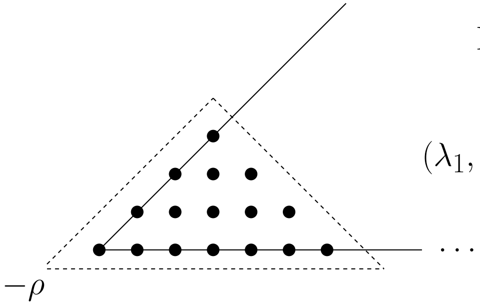


$\Gamma(SO(5))$

$(\lambda_1, \lambda_2) \in \mathbb{Z}^2, \ \lambda_1 \geq \lambda_2 \geq 0$

There are also $\mathbb{Z}$-based quotients of $\mathrm{Gr}(SO(N))$ with only finitely many simples, corresponding to highest weights properly contained in a *shifted Weyl alcove*.

$$\Gamma(SO(5) - O(6))$$



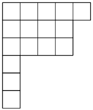
$$(\lambda_1, \lambda_2) \in \mathbb{Z}^2, \ \lambda_1 \geq \lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 \leq 6$$

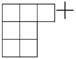
These define fusion rings of type $SO(N) - O(K)$ and $SO(2n+1) - Sp(2k)$. There is a quotient map

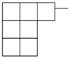
$$\mathrm{Gr}(SO(N)) \to \mathrm{Gr}(SO(N) - G)$$

which takes a simple element to $\pm$ a simple element, or 0. Far from the new bounding hyperplane, the fusion rules are the same as for $\mathrm{Gr}(SO(N))$.

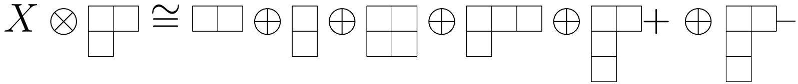
Young diagrams

$$(5,4,4,1,1,1,0,\dots,0) = \quad [5,4^2,1^3] =$$ 

$$(3,2,2) = \quad [3,2^2]^+ =$$ 

$$(3,2,-2) = [3,2^2]^- =$$ 

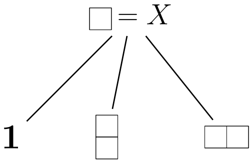
Fundamental fusion rule:
The rule for tensoring with $X \cong [1]$ is adding and removing a box.



The braid element $c_{X,X}$

$SO(N)$-type categories are *tensor generated* by a single simple $X \cong [1]$ (for $N = 3, N \geq 5$). It is self-dual and its tensor square splits into three simples:

$$X^{\otimes 2} \cong \mathbf{1} \oplus [1^2] \oplus [2].$$



$c_{X,X} =$ 

Therefore $c_{X,X} \in \mathrm{End}_{\mathcal{C}}(X^{\otimes 2})$ has three eigenvalues. For a fixed category $\mathcal{C}$, we will always denote

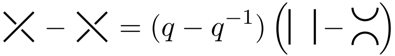
$$q := \text{eigenvalue of } c_{X,X} \text{ on } [2].$$

The classification strategy is to show that the fusion rules and q determine the category $\mathcal{C}$.

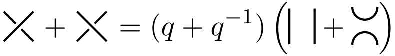
Theorem (Tuba-Wenzl ’03, Morrison-Peters-Snyder ’11)

Let X be a symmetrically self-dual simple object in a ribbon category such that $X^{\otimes 2}$ splits into three simples. Then there is $r \in \mathbb{C}^\times$ such that

With q as above, $c_{X,X}$ satisfies either the Dubrovnik relation:



or Kauffman relation:



Hence $c_{X,X}$ has eigenvalues $(q, -q^{-1}, r^{-1})$ or (q, q^{-1}, r^{-1}).

Classification for Lie type categories

 acts by $(q, -q^{-1})$

$X \otimes X \cong [2] \oplus [1^2]$.

Theorem (Kazhdan-Wenzl, '93)

Any tensor category with $SL(N)$ fusion rules is a twist of **Rep** $SL(N)_q$ *by a 3-cocycle of* $\mathbb{Z}_N$. *Ribbon categories with the fusion rules of* $SL(N)$ *are determined by* q *and equivalent to* **Rep** $SL(N)_q$.

 acts by $(q, \pm q^{-1}, \pm q^{m})$

$X \otimes X \cong [2] \oplus [1^2] \oplus \mathbf{1}$.

Theorem (Tuba-Wenzl, '03)

Ribbon categories with the fusion rules of $O(N)$ *(resp.* $Sp(N)$*) are determined by the eigenvalues of* $c_{X,X}$ *and are equivalent to a twist of* **Rep** $O(N)_q$ *(resp.* **Rep** $Sp(N)_q$*) by a 3-cocycle of* $\mathbb{Z}_2$.

Our main result for $SO(N)$ categories

Let $N \geq 5$ or $N = 3$.

Theorem (C)

Non-symmetric ribbon categories with the fusion rules of $SO(N)$ are determined by the eigenvalues of $c_{X,X}$. Any ribbon category with $SO(2n+1)$ fusion rules and braid eigenvalue q is equivalent to **Rep** $SO_q(2n+1)$*. For $SO(2n)$ every ribbon category is equivalent to a twist of* **Rep** $SO_q(2n)$ *by a 3-cocycle of $\mathbb{Z}_2$.*

- Applies to $SO(N) - O(K)$ and $SO(2n+1) - Sp(K)$ rules
- For $SO(2n+1)$, the braid eigenvalues must be $(q, -q^{-1}, q^{-2n})$ and two categories with q, q' are monoidally equivalent iff $q' \in \{q^{\pm 1}\}$.
- For $SO(2n)$ there are both Dubrovnik and Kauffman cats and two Dubrovnik cats with q, q' are monoidally equivalent iff $q' \in \{\pm q^{\pm 1}\}$.

Proof of $SO(2n+1)$ classification

Proof.

Classically $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}_2$. Hence if $\mathcal{C}$ has $SO(2n+1)$ fusion rules, then

$$\mathcal{C} \boxtimes \textbf{Rep } \mathbb{Z}_2$$

has $O(2n+1)$ fusion rules, so by Tuba-Wenzl it is determined by the eigenvalues of the tensor generator $X \boxtimes -1$. These eigenvalues are the same as the braid eigenvalues for X. Since $\mathcal{C}$ can be recovered from $\mathcal{C} \boxtimes \textbf{Rep } \mathbb{Z}_2$, $\mathcal{C}$ is also determined by the eigenvalues of X. □

- The explicit form of the eigenvalues $(q, -q^{-1}, q^{-2n})$ can be deduced by looking at the q-dim and twist of $\mathbf{1} \boxtimes -1$
- Two $SO(2n+1)$ categories with $q' = -q$ and are not monoidally equivalent, in contrast to $O(2n+1)$.
- There is no $\varepsilon = -1$ family of $SO(2n+1)$ categories.

Monoidal algebras

Suppose $\mathcal{C}$ is a semisimple tensor category. The *monoidal algebra* generated by X is the **strict** monoidal category $\langle X \rangle$ with objects

$$\mathbf{1}, X, X^{\otimes 2}, \dots$$

and hom-spaces coming from $\mathcal{C}$.

Theorem (Kazhdan-Wenzl, Tuba-Wenzl)

If X is a tensor generator of $\mathcal{C}$ then $\mathcal{C}$ can be reconstructed from $\langle X \rangle$ by taking the idempotent completion and adding direct sums.

The diagonal subcategory

The *diagonal* of $\mathcal{C}$ is the monoidal subcategory $\Delta\langle X\rangle$ of $\langle X\rangle$ obtained by setting

$$\mathrm{Hom}_{\Delta\langle X\rangle}(X^{\otimes j}, X^{\otimes k}) = \begin{cases} \mathrm{End}(X^{\otimes k}) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Theorem (Tuba-Wenzl, C)

Suppose $\mathcal{C}$ and $\mathcal{C}'$ are $\mathbb{Z}_2$-graded and tensor generated by X and Y. Then $\Delta\langle X\rangle$ is isomorphic to $\Delta\langle Y\rangle$ if and only if $\mathcal{C}'$ is monoidally equivalent to a twist of $\mathcal{C}$ by a 3-cocycle of $\mathbb{Z}_2$.

Corollary

A $\mathbb{Z}_2$-graded category $\mathcal{C}$ is determined by its diagonal $\Delta\langle X\rangle$ and a sign (choice of 3-cocycle class).

The cocycle construction

Any ribbon category has a mirror (swap braid with its inverse). The mirror category has the same fusion rules.

If $\mathcal{C}$ is also $\mathbb{Z}_2$-graded then there are several other modifications that don't change the fusion rules.

- Twist the associator by a 3-cocycle $\omega \in Z^3(\mathbb{Z}_2, \mathbb{C}^\times)$
- Twist the braiding by an abelian cocycle given by $a : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}^\times$ compatible with ω
- Change the spherical structure with a character of $\mathbb{Z}_2$

Non-trivial ω switches between Dubrovnik and Kauffman categories.

For $\mathcal{C}$ singly-generated there is a unique spherical structure so that every self-dual object is *symmetrically self-dual*.

Data of the diagonal subcategory:

- The semisimple algebras $\mathrm{End}(X^{\otimes k}), k \geq 0$.
- Bilinear maps

$$\mathrm{End}_{\mathcal{C}}(X^{\otimes k}) \times \mathrm{End}_{\mathcal{C}}(X^{\otimes l}) \to \mathrm{End}_{\mathcal{C}}(X^{\otimes k+l})$$
$$(f, g) \mapsto f \otimes g$$

An *isomorphism* of diagonals of $\mathcal{C}$ and $\mathcal{C}'$ is a family $\{\Phi_k\}$ of algebra isomorphisms

$$\Phi_k : \mathrm{End}_{\mathcal{C}}(X^{\otimes k}) \to \mathrm{End}_{\mathcal{C}'}(Y^{\otimes k})$$

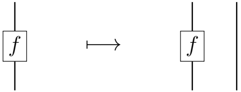
compatible with tensor products.

Strategy for braided $SO(2n)$ classification

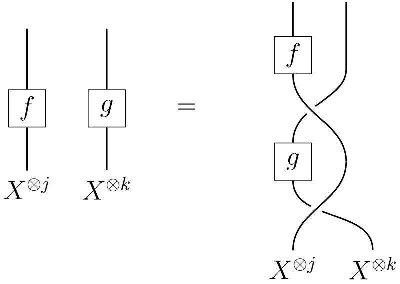
Lemma

If $\mathcal{C}$ is additionally **braided** *then the tensor product maps are determined by the braiding and the inclusions*

$$\ldots \xrightarrow{-\otimes 1} End_{\mathcal{C}}(X^{\otimes k}) \xrightarrow{-\otimes 1} End_{\mathcal{C}}(X^{\otimes k+1}) \xrightarrow{-\otimes 1} \ldots$$



Proof.



$f \otimes 1^{\otimes k}$

braid

$g \otimes 1^{\otimes j}$

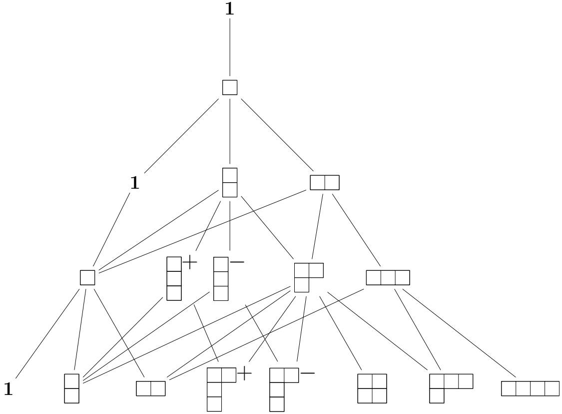
braid

The Bratteli diagram

The *Bratteli diagram* for the inclusions of semisimple algebras

$$\ldots \xrightarrow{-\otimes 1} \mathrm{End}_{\mathcal{C}}(X^{\otimes k}) \xrightarrow{-\otimes 1} \mathrm{End}_{\mathcal{C}}(X^{\otimes k+1}) \xrightarrow{-\otimes 1} \ldots$$

is the same as the *fusion graph* for tensoring with $X \cong [1]$.



Bratteli diagram for $SO(6)$ up to level 4.

Path idempotents and path bases

Since the Bratteli diagram is **multiplicity free** we can define a complete set of minimal idempotents for $\mathrm{End}(X^{\otimes k})$ indexed by paths of length k through the Bratteli diagram:

$$p_S : S = \mathbf{1} \to S(1) \to S(2) \to \cdots \to S(k)$$

- p_S has isotype $S(k)$
- They are compatible with the inclusions $- \otimes 1$:

$$p_S \otimes 1 = \sum_\lambda p_{S \to \lambda}$$

A simple module V^λ for $\mathrm{End}(X^{\otimes k})$ has basis vectors

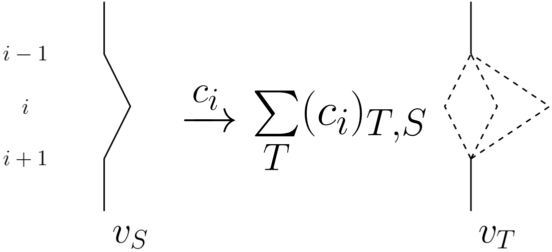
$$\{v_S : S = \mathbf{1} \to S(1) \to \cdots \to S(k-1) \to \lambda\}$$

which are uniquely defined up to scalars by

$$p_S v_T = \delta_{S,T} v_T.$$

The braid elements act locally on a path basis, i.e. if $1 \leq i < k$ then $c_i \in \mathrm{End}(X^{\otimes k})$ and

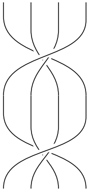
$$c_i v_S \in \mathrm{span}\{v_T \mid T \text{ only differs from } S \text{ at level } i\ \}$$



Our goal is to compute these matrix entries. The diagonal entries are independent of a choice of path basis, while the off-diagonal entries depend on the scaling.

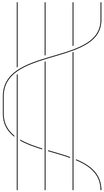
Full twists and Jucys-Murphy elements

The *full twist* Δ_k^2 is defined in $\mathrm{End}(X^{\otimes k})$ using the braiding. It is central in $\mathrm{End}(X^{\otimes k})$.

$\Delta_4^2 =$ 

The *Jucys-Murphy elements* are defined by

$$J_k = \Delta_k^2(\Delta_{k-1}^{-2} \otimes 1) \in \mathrm{End}(X^{\otimes k}).$$

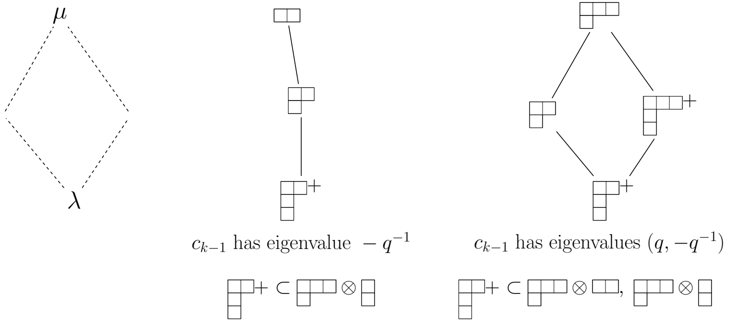
$J_4 =$ 

Lemma

The Jucys-Murphy elements act diagonally in any path basis.

2-level path spaces W_μ^λ

Suppose μ and λ are two levels apart.



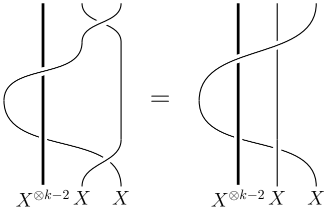
Let

$$W_\mu^\lambda = \text{span}\{w_S : S \text{ is a path from } \mu \to \lambda\}.$$

It supports an action by $J_k, J_{k-1} \otimes 1$ and c_{k-1}. The fusion rules tell us the eigenvalues of c_{k-1}.

If $\lambda \neq \mu$ then $\dim W_\mu^\lambda \leq 2$.

Eigenvalues of JM elements



$$c_{k-1}J_{k-1}c_{k-1} = J_k \qquad (\mathbf{AB}_2)$$

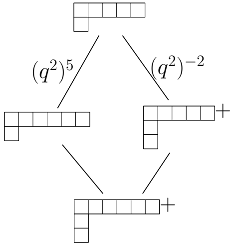
One can write down all 1 and 2-dim diagonalizable matrices which satisfy the Dubrovnik relation, $(\mathbf{AB}_2)$, and the fact $\Delta_k^2 = J_k J_{k-1}$ is central. (c.f. Ariki-Koike '94).

Corollary

The eigenvalues of J_k are determined by the eigenvalues of c_{k-1} and J_{k-1}.

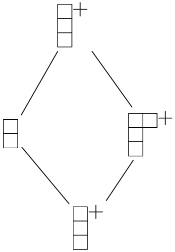
$$(\Delta_k^2)_{S,S} = r^{k-|\lambda|} \sum_{b \in \lambda} q^{2cn(b)}.$$

Restriction of parameters



hooks tell us q^2 is not an lth root of 1

for $l <$ largest hook size

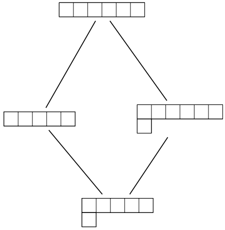


$\lambda, \mu = [1^n]^+$

tells us $r = q^{2n-1}$

$SO(2n) - O(K)$

$\lambda_1 + \lambda_2 \leq K$



$\lambda = [K-1, 1], \ \mu = [K]$

tells us $r = \pm q^{-(K-1)}$

Theorem

If $\mathcal{C}$ is not a fusion category then q is not a root of 1. If $\mathcal{C}$ is a fusion category then q^2 is a primitive $2n + K - 2$-th root of 1. In any case $r = q^{2n-1}$.

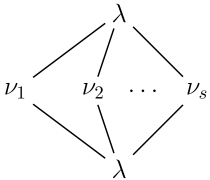
Uniqueness of braid representations

On the "new stuff" we can scale the path basis so c_{k-1} has the matrix

$$c_{k-1} \mapsto \begin{pmatrix} \frac{q^d}{[d]_q} & 1 - \frac{1}{[d]_q^2} \\ 1 & \frac{q^{-d}}{[-d]_q} \end{pmatrix}.$$

On the "old stuff", i.e. W_λ^λ, we can scale the path basis so that $e_{k-1} =$  has the matrix

$$e_{k-1} \mapsto \frac{1}{\dim_{\mathcal{C}} \lambda} \begin{pmatrix} \dim_{\mathcal{C}} \nu_1 & \dots & \dim_{\mathcal{C}} \nu_s \\ \vdots & & \vdots \\ \dim_{\mathcal{C}} \nu_1 & \dots & \dim_{\mathcal{C}} \nu_s \end{pmatrix}$$



Methods of (Leduc-Ram, '97) can be used to show the matrix entries for c_{k-1} are determined by e_{k-1} and JM eigenvalues.

Theorem

The q-dims of every simple object can be expressed as a rational function of q.

Proof of $SO(2n)$ classification theorem

Proof.

Suppose $\mathcal{C}, \mathcal{C}'$ have the same fusion rules and are both Dubrovnik with eigenvalues $(q, -q^{-1}, q^{2n-1})$. Using uniqueness of braid representations we can construct matrix units in $\mathrm{End}(X^{\otimes k})$ (resp. $\mathrm{End}(Y^{\otimes k})$) which are compatible with inclusions and so the braids have the specified matrices.

Then we get algebra isomorphisms $\mathrm{End}(X^{\otimes k}) \to \mathrm{End}(Y^{\otimes k})$ sending matrix units to matrix units. This is compatible with inclusions and braiding so is an isomorphism of diagonals.

By diagonal reconstruction, $\mathcal{C}$ and $\mathcal{C}'$ differ by at most a 3-cocycle twist. However they are both Dubrovnik so they are actually equivalent. □

Open problems

- Description of planar algebra for $SO(N)$ type categories
- Auto-equivalences of $SO(N)$ type categories (Edie-Michell '20)
- Other classification problems: symmetric cases, $SO(4)$, $K \leq 2$, $\mathfrak{so}(N)$, exotic Lie groups
- Computational complexity of braid representations

Thanks for listening!

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[TW05] [KW93] [LR97] [MPS11]